REFERENCES

1. SHIELD R.T., On the plastic flow of metals under conditions of axial symmetry. Proc. Roy. Soc. London. Ser. A., 233, 1193, 1955.
2. COX A.D., EASON C., HOPKINS H.G., Axially symmetric plastic deformations in soils. Phil. Trans. Roy. Sos. London, 254, 1, 1036, 1961.
3. DRUCKER D.C., Limit analysis of two and three dimensional soil mechanics problems. J. Mech. Phys. Solids, 1, 4, 1953.
4. HAYTHORNTWAITE R.M., Range of yield condition in ideal plasticity Proc. Amer. Soc. Civil Engrs. J. Eng. Mech. Div., 127, 1, 1962.
5. HILL R., The Mathematical Theory of Plasticity. Oxford, Clarendon Press, 1956.
6. IVLEV D.D., Theory of Ideal Plasticity. Moscow, Nauka, 1966.
7. KACHANOV L.M., Fundamentals of the Theory of Plasticity. Moscow, Nauka, 1969.
8. TSYTOVICH N.A., Soil Mechanics, Moscow, Vyssh. Shakola, 1979.
9. SHTEIN M.SH., A version of regularizing the equations of ideal plasticity in the case of axial symmetry. PMTF, 3, 1984.
10. COURANT R., Partial Differential Equations. Moscow, Mir, 1964.
11. DRUCKER D.C., PRAGER W. and GREENBERG H.I., Extended limit design theorems for continuous media. Quart. Appl. Math. 9, 4, 1952.
12. FREUDENTHAL A. and GEIRINGER H., Mathematical Theory of an Inelastic Continuum. Moscow, Fizmatgiz, 1962.
13. MOSOLOV P.P. and MYASNIKOV V.P., Mechanics of Rigid-Plastic Media, Moscow, Nauka, 1981.
14. PETROV I.B. and KHOLODOV A.S., On regularizing discontinuous numerical solutions of hyperbolic-type equations. Zh. vychisl. Matematiki i mat. Fiziki, 24, 8, 1984.

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A GENERALIZATION OF THE CANONICAL FORM OF POINCARÉ'S EQUATIONS*

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A class of non-linear reversible replacements of canonical momenta is described, which reduces the Hamiltonian system to a form which differs only slightly from Poincaré's equations /1/ in canonical form, obtained by Chetayev /2/. The difference is solely the fact that the components of the operators which form the right-hand side of the equations of motion may depend on new variables (the Chetayev variables). The usual canonical form of the equations is obtained if the resplacements of the momenta are linear and uniform. Among the important consequences of the equations are Liouville's theorem (on complete integrability), the Kozlov-Kolesnikov theorem (on integrability in integral manifolds) /3/, and the theorem on classes of equivalence of Hamiltonian systems.

1. Initial data and relations. Consider s continuously differentiable functions of the coordinates and canonical momenta

$$y_i = \psi_i(x, p),\ i = 1, \ldots, s \tag{1.1}$$

which are functionally independent and uniquely solvable (in a certain region) in terms of the variables p, i.e., $\det(\partial\psi_i/\partial p_j) \neq 0,\ p_j = \varphi_j(x, y)$ (the functions φ_j, naturally, are not defined everywhere), and generate an s-dimensional Lie algebra ($(\cdot,\cdot)$ are Poisson brackets)

$$(\psi_i, \psi_j) = c_{ij}^k\psi_k\ \ i, j, k = 1, \ldots, s \tag{1.2}$$

Using the operators

$$X_k = \zeta_{x_i}^k \frac{\partial}{\partial x_i} + \zeta_{p_i}^k \frac{\partial}{\partial p_i};\quad \zeta_{x_i}^k = \frac{\partial\psi_k}{\partial p_i},\quad \zeta_{p_i}^k = -\frac{\partial\psi_k}{\partial x_i} \tag{1.3}$$

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the commutation relations (1.2) can be written in the form

$$X_i\psi_j = c_{ij}^k\psi_k = c_{ij}^k y_k \tag{1.4}$$

which can be proved by a direct check.

It is well-known that the operators X_k form a basis of a certain Lie algebra

$$[X_i,\ X_j] = c_{ij}{}^k X_k \tag{1.5}$$

which can also be proved directly on the basis of definition (1.3).

The operators (1.3) act in phase space $\{x, p\}$. We will obtain their form in the space $\{x, y\}$. We will consider an arbitrary differentiable function $F(x, p) = F(x, \varphi(x, y)) = F^*(x, y)$. Obviously

$$X_k F = \left[\zeta_{x_i}^k \left(\frac{\partial F^*}{\partial x_i} + \frac{\partial \psi_j}{\partial x_i}\frac{\partial F^*}{\partial y_j}\right) + \zeta_{p_i}^k \frac{\partial F^*}{\partial y_j}\frac{\partial \psi_j}{\partial p_i}\right]\bigg|_{p_j=\varphi_j} = \left[\zeta_{x_i}^k \frac{\partial}{\partial x_i} + X_k\psi_j \frac{\partial}{\partial y_j}\right]\bigg|_{p_j=\varphi_j} \cdot F^*$$

Bearing in mind Eq.(1.4) we obtain

$$X_k F = X_k^* F^*, \quad X_k^* = \xi_i{}^k(x, y)\frac{\partial}{\partial x_i} + c_{ki}^{\gamma} y_\gamma \frac{\partial}{\partial y_i}$$

$$(\xi_i{}^k(x, y) = \zeta_{x_i}^k\big|_{p_i=\varphi_i})$$

In accordance with (1.5) we have the commutation relations

$$[X_i^*, X_j^*] = c_{ij}^k X_k^* \tag{1.6}$$

2. The equations of motion.

Consider the Hamiltonian system

$$x_i^{\cdot} = \frac{\partial H}{\partial p_i}, \quad p_i^{\cdot} = -\frac{\partial H}{\partial x_i}, \quad H = H(x, p) \tag{2.1}$$

We have

$$x_i^{\cdot} = \frac{\partial H}{\partial p_i} = \frac{\partial \psi_j}{\partial p_i}\bigg|_{p_j=\varphi_j} \cdot \frac{\partial H^*}{\partial y_j} = \xi_i{}^j(x, y)\frac{\partial H^*}{\partial y_j}$$

$$y_i^{\cdot} = \frac{\partial \psi_i}{\partial x_j} x_j^{\cdot} + \frac{\partial \psi_i}{\partial p_j} p_j^{\cdot} = \frac{\partial \psi_i}{\partial x_j}\frac{\partial H}{\partial p_j} - \frac{\partial \psi_i}{\partial p_j}\frac{\partial H}{\partial x_j} = -X_i H = -X_i^* H^*$$

$$(H^*(x, y) = H(x, \varphi(x, y)) = H(x, p))$$

Hence, the equations of motion take the form

$$x_i^{\cdot} = Y_i^* H^*, \quad y_i^{\cdot} = -X_i^* H^*, \quad Y_i^* = \xi_i{}^j(x, y)\,\partial/\partial y_j \tag{2.2}$$

When $\xi_i{}^j = \delta_i{}^j$ we revert to the Hamiltonian system (2.1). If the functions ψ_i are linear and uniform in the moments $y_i = \xi_j{}^i(x)\,p_j$, Eqs.(2.2) reduces to the canonical form of the Poincaré-Chetayev equations. System (2.2) largely preserves the features of this classical form. We will briefly consider only the most important property of the shift operator.

By grouping terms, as in /4/, it can be established that the operator of differentiation with respect to time along the trajectories of the system (2.2) can take the form

$$\frac{d}{dt} \equiv S = -\frac{\partial H^*}{\partial x_j} Y_j^* + \frac{\partial H^*}{\partial y_j} X_j^*, \quad j = 1, \ldots, s \tag{2.3}$$

in the case of its action on the function specified in the space $\{x, y\}$, and

$$S = \frac{\partial}{\partial t} - \frac{\partial H^*}{\partial x_j} Y_j^* + \frac{\partial H^*}{\partial y_j} X_j^* \tag{2.4}$$

in the case of action on the function specified in the extended space $\{t, x, y\}$.

The system of operators X_j^*, Y_j^* is independent and closed. It has a simple multiplication table: apart from (1.6) it contains the commutation relations

$$[Y_k^*,\ Y_l^*] = 0, \quad k,\ l = 1, \ldots, s \tag{2.5}$$

$$[X_k^*, Y_l^*] = \frac{\partial \xi_l{}^k}{\partial x_i} Y_i^* - \frac{\partial \xi_l{}^k}{\partial y_i} X_i^* \tag{2.6}$$

We will prove these by using the obvious identities

$$\frac{\partial \varphi_\gamma}{\partial y_i}\frac{\partial \psi_i}{\partial p_k} = \delta_k{}^\gamma$$

In fact

$$[Y_k^*, Y_l^*] = (Y_k^*\xi_l{}^j - Y_l^*\xi_k{}^j)\frac{\partial}{\partial y_j}$$

$$Y_k{}^*\xi_l{}^j - Y_l{}^*\xi_k{}^j = \left[\xi^i_{x_k}\frac{\partial \xi^j_{x_l}}{\partial p_\gamma}\frac{\partial \varphi_\gamma}{\partial y_i} - \xi^i_{x_l}\frac{\partial \xi^j_{x_k}}{\partial p_\gamma}\frac{\partial \varphi_\gamma}{\partial y_i}\right]\Bigg|_{p_j=\varphi_j} =$$
$$\left[\frac{\partial \psi_i}{\partial p_k}\frac{\partial}{\partial p_\gamma}\left(\frac{\partial \varphi_j}{\partial p_l}\right)\frac{\partial \varphi_\gamma}{\partial y_i} - \frac{\partial \psi_i}{\partial p_l}\frac{\partial}{\partial p_\gamma}\left(\frac{\partial \psi_j}{\partial p_k}\right)\frac{\partial \varphi_\gamma}{\partial y_i}\right]\Bigg|_{p_j=\varphi_j} =$$
$$\left[\frac{\partial}{\partial p_k}\left(\frac{\partial \psi_j}{\partial p_l}\right) - \frac{\partial}{\partial p_l}\left(\frac{\partial \psi_j}{\partial p_k}\right)\right]\Bigg|_{p_j=\varphi_j} = 0$$

Relations (2.5) are proved. We will now prove (2.6). We have

$$[X_k{}^*, Y_l{}^*] = -Y_l{}^*\xi_k{}^j\frac{\partial}{\partial x_j} + (X_k{}^*\xi_l{}^j - Y_l{}^* c^\gamma_{kj} y_\gamma)\frac{\partial}{\partial y_j}$$

Using (2.5) we obtain

$$-Y_l{}^*\xi_j{}^k\frac{\partial}{\partial x_j} = -Y_j{}^*\xi_l{}^k\frac{\partial}{\partial x_j} = -\frac{\partial \xi_l{}^k}{\partial y_i}\left(\xi_j{}^i\frac{\partial}{\partial x_j}\right)$$

Further

$$X_k{}^*\xi_l{}^j - Y_l{}^* c^\gamma_{kj} y_\gamma = X_k{}^*\xi_l{}^j - c^\gamma_{kj}\,\xi_l{}^i\frac{\partial y_\gamma}{\partial y_i} = X_k{}^*\xi_l{}^j - c^\gamma_{kj}\xi_l{}^\gamma = X_j{}^*\xi_l{}^k$$

Hence

$$[X_k{}^*, Y_l{}^*] = -\frac{\partial \xi_l{}^k}{\partial y_i}\left(\xi_j{}^i\frac{\partial}{\partial x_j}\right) + \left(\xi_i{}^j\frac{\partial \xi_l{}^k}{\partial x_i} + c^\gamma_{ji} y_\gamma\frac{\partial \xi_l{}^k}{\partial y_i}\right)\frac{\partial}{\partial y_j} =$$
$$\frac{\partial \xi_l{}^k}{\partial x_i}\left(\xi_i{}^j\frac{\partial}{\partial y_j}\right) - \frac{\partial \xi_l{}^k}{\partial y_i}\left(\xi_j{}^i\frac{\partial}{\partial x_j} + c^\gamma_{ij} y_\gamma\frac{\partial}{\partial y_j}\right)$$

which is identical with relations (2.6).

It follows from (1.8) and (2.6) that if the Hamiltonian H^* and the functions $\xi_1{}^k, \ldots, \xi^k_{s-1}$ $(k = 1, \ldots, s)$ do not depend on the variable x_s, the system of operators $X_1{}^*, \ldots, X_s{}^*, Y_1{}^*, \ldots, Y^*_{s-1}$ is closed; then the system of equations

$$X_1{}^*\omega = \ldots = X_s{}^*\omega = Y_1{}^*\omega = \ldots = Y^*_{s-1}\omega = 0$$

is consistent. It follows from Eq.(2.4) for the shift operator that a unique solution ω of this system is the integral of Eqs.(2.2). It is (in a certain sense) a non-linear analogue of the cyclic integral corresponding to the coordinate x_s.

3. The immediate consequences of the equations of motion.

We will consider one of the important special cases when $\psi_i(x, p) = c_1, \ldots, \psi_s(x, p) = c_s$ are the first integrals of motion of system (2.1).

We obtain from the second subsystem of Eqs.(2.2)

$$X_i{}^*H^*|_{y_j=c_j} = 0 \tag{3.1}$$

The change in the coordinates x_i will be described in this case by the first subsystem of Eqs.(2.2)

$$x_i{}^{\cdot} = Y_i{}^*H|_{y_j=c_j} = \xi_i{}^j(x, c)\frac{\partial H^*(x, c)}{\partial c_j} \tag{3.2}$$

According to the commutation relations (1.8), the operators

$$X_k{}^* = \xi_i{}^k(x, c)\frac{\partial}{\partial x_i} + c^\gamma_{ki} c_\gamma\frac{\partial}{\partial c_i} \tag{3.3}$$

form a basis of a Lie algebra, to which there corresponds a local group of transformations G which act in the space $\{x, c\}$. We will show that the transformations of the group G transform Eqs.(3.2) into the equations

$$x_i'^{\cdot} = \xi_i{}^j(x', c')\,\partial H^*(x', c')/\partial c_j' \tag{3.4}$$

To do this consider the shift operator along the trajectories of system (3.2)

$$S_1 = \frac{\partial}{\partial t} + \xi_i{}^j(x, c)\frac{\partial H^*(x, c)}{\partial c_j}\frac{\partial}{\partial x_i}$$

Taking relations (2.6) and (3.1) into account, we obtain

$$[S_1, X_k{}^*] = (S_1\xi_i{}^k - X_k{}^*Y_i{}^*H^*)|_{y_j=c_j}\cdot\frac{\partial}{\partial x_i} = \left[Y_j{}^*H^*\frac{\partial \xi_i{}^k}{\partial x_j} - Y_i{}^*X_k{}^*H^* +\right.$$
$$\left.\frac{\partial \xi_i{}^k}{\partial c_j}X_j{}^*H^* - \frac{\partial \xi_i{}^k}{\partial x_j}Y_j{}^*H^*\right]\Bigg|_{y_j=c_j}\cdot\frac{\partial}{\partial x_i} = 0, \quad k = 1, \ldots, s$$

1°. It follows from the form of the operators (3.3) that when $c_{ij}{}^k = 0$, the group G commutes, and is a symmetry group of system (3.2). According to (1.2), $\psi_i = c_i$ are the first integrals in the involution of Eqs.(2.2). The system of equations

$$S_1\omega_i = 0,\ X_k{}^*\omega_i = \delta_k{}^i;\ i, k = 1, \ldots, s \tag{3.5}$$

is consistent for each $i = 1, \ldots, s$. It defines the first s integrals of system (3.2) is quadratures. This is Liouville's theorem on the complete integrability of Hamiltonian systems. The quadratures are obtained from the formulas

$$\omega_i = tf_0^{(i)}(x, c) + \int f_j^{(i)}(x, c)\, dx_j = k_i = \text{const}$$

Here $f_0{}^{(i)} = \partial\omega_i/\partial t,\ f_j{}^{(i)} = \partial\omega_i/\partial x_j$ is the result of the solution of system (3.5) relative to the derivatives of ω_i (solvability occurs since $\det(\xi_j{}^i) \neq 0$).

2°. If the group G does not commute, but is solvable (/5/, p.208), but the constants of integration c_j are constrained by the conditions

$$c_{ij}{}^k c_k = 0 \tag{3.6}$$

then, according to the form of the opeators (3.3), the group G acts on the set (3.6) as a symmetry group of Eqs.(3.2). From the well-known Lie theorem, system (3.2) can be integrated in quadratures. This is the Kozlov-Kolesnikov theorem.

3°. We will now consider the case when conditions (3.6) are not satisfied. Suppose R^s is a Euclidean space of constants c_i. There is a point $c = (c_1, \ldots, c_s) \in R^s$ corresponding to each fixed set of these constants. We will define the region Q as the set of all points of space $\{x, c\}$, in which the condition $\det(\xi_j{}^i) \neq 0$ is satisfied. Henceforth we will assume that at each point considered $c \in R^s$ the variables x do not go outside the limits of Q. Then, the effectiveness of the action in R^s of the group G will depend on the rank of the matrix $K_s = (c_{ij}{}^\gamma c_\gamma)$.

Thus, if $\det K_s \neq 0$ (which is only possible for even s), the action of the group will be locally transitive. This means that however close the points $c \in R^s$ and $c' \in R^s$ are to one another we can specify a continuous (or even smooth) transformation $g \in G$, which converts the corresponding systems (3.2) and (3.4) into one another. The phase portraits of these systems are therefore topologically (or smoothly) equivalent.

We will now carry out a more detailed analysis. We will denote by Γ_μ the maximally wide region in which all the minors of the matrices K_s of order μ vanish. In the sequence $\Gamma_1, \Gamma_2, \ldots, \Gamma_s, R^s$ each of the regions Γ_μ is either contained in the nearest next one $\Gamma_\mu \subset \Gamma_{\mu+1}$ or coincides with it: $\Gamma_\mu = \Gamma_{\mu+1}$. For even s, generally speaking, $\Gamma_s \subset R^s$, and for odd s $\det K_s = 0$ and, consequently, $\Gamma_s = R^s$. In the chain of imbeddings which define the sequence $\Gamma_1, \ldots, R^s$, the most typical branch has the form $\Gamma_\mu \subset \Gamma_{\mu+1} = \ldots = \Gamma_{\mu+\nu} \subset \Gamma_{\mu+\nu+1}$. Obviously in the region $\Gamma_{\mu+\nu}$ the system of equations

$$c_{ij}{}^\gamma c_\gamma \partial\Omega/\partial c_j = 0,\ i = 1, \ldots, s \tag{3.7}$$

has a general rank μ and has $\nu - 1$ functionally independent solutions

$$\Omega_1(c) = I_1, \ldots, \Omega_{\nu-1}(c) = I_{\nu-1} \tag{3.8}$$

which are invariants of the action of the group G in the space R^s. For $s = 3$, for example, there is one invariant (3.8) for each non-commuting group G. All these have been calculated in explicit form (/4/, p.52).

At the intersection of the set (3.8) for fixed numerical values of $I_1, \ldots, I_{\nu-1}$ with the region $\Gamma_{\mu+\nu} \setminus \Gamma_\mu$ the transformations of the group G act locally transitively.

Hence, we have the following theorem.

Theorem. For fairly close points $c, c' \in \Gamma_{\mu+\nu} \setminus \Gamma_\mu$, corresponding to the same numerical values of the invariants $I_1, \ldots, I_{\nu-1}$, the phase portraits of system (3.2) and (3.4) are continuously (or smoothly) equivalent to the region Q.

The case of Euler motion of a solid is a clear illustration of this theorem.

We can take as the variables y_1, y_2, and y_3 the constant projections of the kinetic momentum on fixed axes. The equivalence of the phase portraits for the same value of k of the kinetic momentum is realized by a group of rotations. The presence of this group, in fact, is also usually employed when, in order to simplify the problem, a special choice of fixed axes is made from the origin itself: $y_1 = y_2 = 0,\ y_3 = k$.

The situation may not be quite so simple for other mechanical problems.

REFERENCES

1. POINCARE H., Sur une forme nouvelle des équations de la mecanique. C. r. Acad. sci, 132, 1901.
2. CHETAYEV N.G., The Stability of Motion. Papers on Analytical Mechanics, Moscow, Izd. Akad. Nauk SSSR, 1901
3. KOZLOV V.V. and KOLESNIKOV N.N., The integrability of Hamiltonian systems, Vestnik MGU. Ser. Mat. Makh. 6, 1979.

4. NARKASHOV L.M., The Poincaré and Poincaré-Chetayev equations, PMM, 49, 1, 1985.
5. CHEBOTAREV N.G., The Theory of Lie Groups, Gostekhteorizdat, Moscow, 1940.

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THE PROBLEM OF THE DIFFRACTION OF INTERNAL WAVES AT THE EDGE OF A SEMI-INFINITE FILM*

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In a continuation of the research described in /1-4/ on the diffraction of waves, described by the Klein-Gordon equation, the diffraction of external waves at the boundary of a semi-infinity film situated on the surface of a stratified liquid is considered. Among the many papers devoted to the scattering of acoustic waves by rectilinear objects we mention /5-7/. The need to take into account the properties of the surface covering the liquid led to a study of the boundary value problem for the Helmholtz equation with boundary conditions containing higher-order derivatives than the equation itself. Consideration of the surface tension of a semi-infinite film leads to a similar situation. However, in this case the propagation of the waves is described by an equation of the hyperbolic and not the elliptic type.

1. To study two-dimensional motions of an incompatible ideal liquid we will introduce a Cartesian system of coordinates $\{x, O, z\}$. Consider an infinite plane layer $Q = \{(x, z): -\infty < x < \infty, -h < z < 0\}$ of a stratified liquid, bounded from below (for $z = -h$) by a solid bottom. Above (where $z = 0$) the boundary of the liquid consists of two parts; for $x < 0$ the surface of the liquid is free, and for $x \geqslant 0$ the liquid is covered by a thin film having a surface tension σ. The density of the liquid in the unperturbed state has the distribution $\rho_0(z) = \rho_0 e^{-2\beta z}$, $\beta > 0$.

The small oscillations of the liquid are described by the following system of equations /8/:

$$\rho_0(z)\partial \mathbf{V}/\partial t + \nabla p + \mathbf{e}_z\rho_1 g = 0$$
$$\partial/\partial t\rho_1 + (\mathbf{e}_z, \mathbf{V})\rho_0'(z) = 0, \ \operatorname{div}\mathbf{V} = 0 \tag{1.1}$$

where $\mathbf{V} = \{v_1, v_2\}$ is the vector of the velocity of the liquid particles, ρ_1 is the change in the density due to motions of the liquid, p is the dynamic pressure, $\mathbf{e}_z$ is the unit vector of the Oz axis, and g is the acceleration due to gravity.

If we introduce the stream function Ψ using the formulas $v_1 = \Psi_z$, $v_2 = -\Psi_x$ and then the function $u = \Psi e^{-\beta z}$, the integration of system (1.1) can be reduced to solving the equation

$$\partial^2/\partial t^2 [\Delta_2 u - \beta^2 u] + \omega_0^2 u_{xx} = 0 \tag{1.2}$$

where Δ_2 is the Laplace operator with respect to x and z and $\omega_0^2 = 2\beta g$ is the square of the Brent-Viaisial frequency.

For steady-state wave motion, which depends on time as $e^{-i\omega t}$, and $\omega < \omega_0$, Eq.(1.2) can be written as the Klein-Gordon equation

$$u_{zz} - \beta^2 u = \frac{1}{a^2} u_{xx}, \quad \frac{1}{a^2} = \frac{\omega_0^2}{\omega^2} - 1 > 0 \tag{1.3}$$

The condition for the solid bottom to be impenetrable and the boundary condition on the free surface /2/ have the form

$$u = 0, \ z = -h, \ x \in R^1 \tag{1.4}$$
$$u_z + \beta u + (g/\omega^2) u_{xx} = 0, \ z = 0, \ x < 0 \tag{1.5}$$

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